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A NEW SPECIAL FUNCTION ADVANCED MODIFIED M- FUNCTION AND FRACTIONAL CALCULUS

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ABSTRACT

In recent year's many special functions given by mathematicians, here a new function termed as Advanced Modified M- function has been introduced. This Function is a particular case of H-function given in [2,3]. This function is important because hypergeometric function and Mittag-Leffler function follow as particular cases and these functions have great significance in the context of problems in physics, biology, engineering and applied sciences. It is to be noted that Mittag-Leffler [4,5] function occurs as solution of fractional integral equations in those subjects. In this paper we have also obtained the fractional integration and fractional differentiation of Advanced Modified M-function.

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KEYWORDS: Fractional Calculus, Advanced Modified M-   function, Riemann-Liouville Operator..

INTRODUCTION

We give the new special function, called **Advanced Modified M − function**, which is the most generalization of New Generalized Mittag-Leffler Function . Here, we give first the notation and the definition of the New Special **Advanced Modified M −   function**, introduced by the author as follows:

$$\,_p^{\alpha,\beta,\gamma,\delta,\rho}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}(t) = \sum_{n=o}^{\infty} \frac{(a_1)_n \;\dots\; (a_p)_n\,(\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{\prod_{i=1}^n a_i^{a_i}\;\; (ct)^{(n+\gamma)\alpha-\beta-1}}{\prod_{i=1}^n b_i^{b_i}\;\; n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} \qquad (1.1)$$

RELATIONSHIP OF THE $\,_p^{\alpha,\beta,\gamma,\delta,\rho}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}$ FUNCTION AND OTHER SPECIAL FUNCTIONS

In this section, we define relationship of **M** function and various special functions

(1) If $\dfrac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}} = 1$ Then (1.1) **Advanced Modified M −   function** converts into M-function, introduced by the author as follows:

$$\,_p^{\alpha,\beta,\gamma,\delta,\rho}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}(t) = \sum_{n=o}^{\infty} \frac{(a_1)_n \;\dots\; (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{(ct)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} \qquad (2.1)$$

There are p upper parameters $a_1, a_2, \dots a_p$ and q lower parameters $b_1,\; b_2, \dots b_q$,$\alpha,\; \beta,\; \gamma, \delta, \rho \in C, Re(\alpha) > 0,\; Re(\beta) > 0,\; Re(\gamma) > 0,\; Re(\delta) > 0,\; Re(\rho) > 0,\; Re(\alpha\gamma - \beta) > 0$ and $(a_j)_k\,(b_j)_k$ are pochammer symbols and $k_1, \dots k_p$, $l_1, \dots l_q$ are constants. The function (1) is defined when none of the denominator parameters $b_j s,\; j = 1,2,\dots q$ is a negative integer or zero. If any parameter a_j is negative then the function (1) terminates into a polynomial in (t).

(2). For $k_1 = a, k_2 \ldots k_p = 1, l_1, \ldots l_q = 1, \delta = 1$ and $\rho = 1, c = 1$, we defined relationship of **Advanced Modified M − function** and various special functions. .

The **Advanced Modified M − function** reduces to New Generalized Mittag-Leffler Function [6]

$$^{\alpha,\beta,\gamma,1,1}_{\quad\;\; 1}\mathbf{M}^{a,1}_1(t) = t^{\alpha\gamma-\beta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n (a)^n (t)^{\alpha n}}{n!\,\Gamma\big((n+\gamma)\alpha-\beta\big)} = t^{\alpha\gamma-\beta-1} E^{\gamma}_{\alpha,\alpha\gamma-\beta}[at^\alpha] \tag{2}$$

(ii). We take $\gamma = 1$, in (2) obtained Generalized Mittag-Leffler Function [10], we get

$$^{\alpha,\beta,1,1,1}_{\qquad 1}\mathbf{M}^{a,1}_1(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{(n+1)\alpha-\beta-1}}{\Gamma\big((n+1)\alpha-\beta\big)} = t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}[at^\alpha] \tag{3}$$

(iii). Further $\beta = \alpha - 1$,, in(3), this **M** function converts Mittag-Leffler Function [6,7], we have

$$^{\alpha,\alpha-1,1,1,1}_{\qquad\quad 1}\mathbf{M}^{a,1}_1(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{n\alpha}}{\Gamma(n\alpha+1)} = E_\alpha[at^\alpha] \tag{4}$$

(iv). When $a = 1$ and $\beta = \alpha - \beta$ in (4) then the **M** function treats as Agarwal's Function [1]

$$^{\alpha,\alpha-\beta,1,1,1}_{\qquad\quad 1}\mathcal{M}^{1,1}_1(t) = \sum_{n=0}^{\infty} \frac{(t)^{n\alpha+\beta-1}}{\Gamma(n\alpha+\beta)} = E_{\alpha,\beta}[t^\alpha] \tag{5}$$

(ix). On substituting $\alpha = 1, \beta = -\beta$ in (3), we get Miller and Ross Function [5].

$$^{1,-\beta,1,1,1}_{\qquad 1}\mathcal{M}^{a,1}_1(t) = \sum_{n=0}^{\infty} \frac{(a)^n (t)^{n+\beta}}{\Gamma(n+\beta+1)} = E_t[\beta, a] \tag{6}$$

MATHEMATICAL PREREQSITIES

The Riemann-Liouville fractional integral of order $v \in C$ is defined by Miller and Ross (1993, p.45)

$$_0 D_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-u)^{v-1} f(u)\,du, \tag{3.1}$$

where Re(v)>0. Following Samko et al. (1993, p. 37) we define the fractional derivative for $\alpha > 0$ in the form

$$_0 D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)\,du}{(t-u)^{\alpha-n+1}}, \qquad (n = [\text{Re}(\alpha)] + 1), \tag{3.2}$$

Where $[\text{Re}(\alpha)]$ means the integral part of $\text{Re}(\alpha)$.

FRACTIONAL INTEGRAL AND FRACTIONAL DIFFERENTIAL OF THE Advanced Modified M − FUNCTION

Let us consider the fractional Riemann − Liouville (R-L) integral operator, as in [5,8] (for lower limit $a = 0$ with respect to variable z) of the **Advanced Modified M − function** (1).

$$_0D_t^{-v}{}^{\alpha,\beta,\gamma,\delta,\rho}_p\mathbf{M}_q^{k_1,\ldots k_p,l_1,\ldots l_q;c}(t)$$

$$= \frac{1}{\Gamma(v)}\int_0^z (z-t)^{v-1}\sum_{n=o}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\frac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}}\frac{(ct)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma((n+\gamma)\alpha-\beta)}\,dt$$

$$= \frac{1}{\Gamma(v)}\sum_{n=o}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma((n+\gamma)\alpha-\beta)}\times$$

$$\int_0^z (z-t)^{v-1}(t)^{(n+\gamma)\alpha-\beta-1}\,dt$$

$$= \frac{1}{\Gamma(v)}\sum_{n=o}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}}\frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma((n+\gamma)\alpha-\beta)}z^{(n+\gamma)\alpha-\beta+v-1}\times$$

$$B((n+\gamma)\alpha-\beta,v)$$

$$= \frac{1}{\Gamma(v)}\sum_{n=o}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}}\frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma((n+\gamma)\alpha-\beta)}z^{(n+\gamma)\alpha-\beta+v-1}\times$$

$$\frac{\Gamma((n+\gamma)\alpha-\beta)\Gamma(v)}{\Gamma((n+\gamma)\alpha-\beta+v)}$$

$$= \frac{\Gamma((n+\gamma)\alpha-\beta-n)}{\Gamma((n+\gamma)\alpha-\beta+v-n)}z^v\sum_{n=o}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_n}{(b_1)_n\ldots(b_q)_n}\frac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}}\frac{(\delta)_n}{(\rho)_n}\frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma((n+\gamma)\alpha-\beta)}$$

$$\times z^{(n+\gamma)\alpha-\beta-1}\frac{((n+\gamma)\alpha-\beta-n)_n}{((n+\gamma)\alpha-\beta+v-n)_n}$$

$$= \frac{\Gamma((n+\gamma)\alpha-\beta-n)}{\Gamma((n+\gamma)\alpha-\beta+v-n)}z^v\sum_{n=o}^{\infty}\frac{(a_1)_n\ldots(a_p)_n(\gamma)_n}{(b_1)_n\ldots(b_q)_n}\frac{(\delta)_n}{(\rho)_n}\frac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}}\frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\frac{(c)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma((n+\gamma)\alpha-\beta)}$$

$$z^{(n+\gamma)\alpha-\beta-1}\frac{((n+\gamma)\alpha-\beta-n)_n}{((n+\gamma)\alpha-\beta+v-n)_n}$$

$$_0D_t^{-v\ \alpha,\beta,\gamma,\delta,\rho}{}_p\mathbf{M}_q^{k_1,\ldots k_p,l_1,\ldots l_q;c}(t) = \frac{\Gamma((n+\gamma)\alpha-\beta-n)}{\Gamma((n+\gamma)\alpha-\beta+v-n)}\ z^{v\ \alpha,\beta,\gamma,\delta,\rho}{}_p\mathbf{M}_q^{k_1,\ldots k_p,l_1,\ldots l_q;c}$$

$$(a_1\ldots a_p,\gamma,\delta,\left(((n+\gamma)\alpha-\beta-n)\ \right);,b_1\ \ldots\ b_q,\rho,,((n+\gamma)\alpha-\beta+v-n);z)\ \ (4.1)$$

Riemann $-$ Liouville Fractional derivative of **Advanced Modified M$-$ function** which indices p,q are increased to $(p+1),(q+1)$.

Analogously, Riemann $-$ Liouville fractional derivative operator [5,8] of the **M** -Function with respect to z.

$$D_z^{v\ \alpha,\beta,\gamma,\delta,\rho}{}_p\mathbf{M}_q^{k_1,\ldots k_p,l_1,\ldots l_q;c}(t) = \frac{1}{\Gamma(n-v)}\left(\frac{d}{dz}\right)^n\int_0^z (z-t)^{n-v-1\ \alpha,\beta,\gamma,\delta,\rho}{}_p\mathbf{M}_q^{k_1,\ldots k_p,l_1,\ldots l_q;c}(t)\ dt$$

$$D_z^{v\ \alpha,\beta,\gamma,\delta,\rho}{}_p\mathbf{M}_q^{k_1,\ldots k_p,l_1,\ldots l_q;c}(t) = \frac{1}{\Gamma(n-v)}\left(\frac{d}{dz}\right)^n\int_0^z (z-t)^{n-v-1}$$

$$\sum_{n=o}^{\infty}\frac{(a_1)_n\ \ldots\ (a_p)_n\ (\gamma)_n}{(b_1)_n\ldots(b_q)_n}\ \frac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}}\ \frac{(\delta)_n}{(\rho)_n}\ \frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\ \frac{(ct)^{(n+\gamma)\alpha-\beta-1}}{n!\,\Gamma((n+\gamma)\alpha-\beta)}\ dt$$

$$= \frac{1}{\Gamma(n-v)}\sum_{n=o}^{\infty}\frac{(a_1)_n\ \ldots\ (a_p)_n\ (\gamma)_n}{(b_1)_n\ldots(b_q)_n}\ \frac{(\delta)_n}{(\rho)_n}\ \frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\ \frac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}}\ \frac{1}{n!\,\Gamma((n+\gamma)\alpha-\beta)}\ c^{(n+\gamma)\alpha-\beta-1}\left(\frac{d}{dz}\right)^n$$

$$\int_0^z (z-t)^{n-v-1}\ t^{(n+\gamma)\alpha-\beta-1}\ dt$$

$$= \frac{1}{\Gamma(n-v)}\sum_{n=o}^{\infty}\frac{(a_1)_n\ \ldots\ (a_p)_n\ (\gamma)_n}{(b_1)_n\ldots(b_q)_n}\ \frac{\prod_{i=1}^n a_i^{a_i}}{\prod_{i=1}^n b_i^{b_i}}\ \frac{(\delta)_n}{(\rho)_n}\ \frac{k_1^n\ldots k_p^n}{l_1^n\ldots l_q^n}\ n!\,\Gamma((n+\gamma)\alpha$$

$$-\beta)\ c^{(n+\gamma)\alpha-\beta-1}\left(\frac{d}{dz}\right)^n$$

$$z^{(n+\gamma)\alpha-\beta+n-v-1}\ B(n-v,(n+\gamma)\alpha-\beta)$$

We use the modified Beta-function:

$$\int_a^b (b-t)^{\beta-1}\ (t-a)^{\alpha-1}\ dt\ = (b-a)^{\alpha+\beta-1}\ B(\alpha,\beta),\ \ \text{for } R(\alpha)>0, R(\beta)>0$$

$$= \frac{1}{\Gamma(n-v)} \sum_{n=o}^{\infty} \frac{(a_1)_n \, \dots \, (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{(\delta)_n}{(\rho)_n} \frac{\prod_{i=1}^{n} a_i^{\,a_i}}{\prod_{i=1}^{n} b_i^{\,b_i}} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} n! \, \Gamma((n+\gamma)\alpha$$

$$- \beta) \, c^{(n+\gamma)\alpha-\beta-1} \left(\frac{d}{dz}\right)^n z^{(n+\gamma)\alpha-\beta+n-v-1}$$

$$\frac{\Gamma(n-v)\Gamma((n+\gamma)\alpha-\beta)}{\Gamma((n+\gamma)\alpha-\beta+n-v)} \quad (4.2)$$

Where $k+1>0, n-v>0$

Differentiation n times the term $z^{(n+\gamma)\alpha-\beta+n-v-1}$ and using again

$\Gamma(a+k) = (a)_k \Gamma(a)$, representation (4.2) reduces to

$$= \frac{1}{\Gamma(\alpha(n+\gamma)-\beta+n)} z^{-v} \sum_{n=o}^{\infty} \frac{(a_1)_n \, \dots \, (a_p)_n (\gamma)_n}{(b_1)_n \dots (b_q)_n} \frac{\prod_{i=1}^{n} a_i^{\,a_i}}{\prod_{i=1}^{n} b_i^{\,b_i}} \frac{(\delta)_n}{(\rho)_n} \frac{k_1^n \dots k_p^n}{l_1^n \dots l_q^n} \frac{1}{n!}$$

$$\times (cz)^{(n+\gamma)\alpha-\beta-1}(\alpha(n+\gamma)-\beta)_n \frac{1}{((n+\gamma)\alpha-\beta-v)_n}$$

$$D_z^{v}\,{}^{\alpha,\beta,\gamma,\delta,\rho}_{p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}$$

$$\frac{1}{\Gamma(\alpha(n+\gamma)-\beta+n)} z^{-v}\,{}^{\alpha,\beta,\gamma,\delta,\rho}_{p}\mathbf{M}_q^{k_1,\dots k_p,l_1,\dots l_q;c}\left(a_1 \dots a_p, \gamma, \delta, \left(\left((n+\gamma)\alpha-\beta\right)\right); , b_1 \dots b_{q,}\, \rho, , ((n+\gamma)\alpha-\beta-v); \, z\right) \quad (4.3)$$

(4.3) gives a Riemann − Liouville fractional derivative of **Advanced Modified M − function**, which indices p, q are increased to (p+1),(q+1).

CONCLUSION

In this present work, we have introduced a new special function and two new results in which we have obtained fractional Integration and Fractional Differentiation of **Advanced Modified M − Function** .

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